



# Participation constraints in first-price auctions

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## Abstract

We study the endogenous participation problem when bidders are characterized by a two-dimensional private information on valuations and participation costs in first-price auctions. Bidders participate whenever their private costs are less than or equal to the expected revenue from participating. We show that there always exists an equilibrium in this general setting with two-dimensional types of ex-ante heterogeneous bidders. When bidders are ex-ante homogeneous, there is a unique symmetric equilibrium, but asymmetric equilibria may also exist. We provide conditions under which the equilibrium is unique (not only among symmetric ones). In the symmetric equilibrium, we show that the equilibrium cutoff of participation costs described above which bidders never participate, is lower when the distribution of participation costs is first-order stochastically dominated.

**Keywords** Existence and uniqueness · Two-dimensional private information · First-price auctions · Participation costs

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## 1 Introduction

Auctions are an efficient way to allocate scarce resources. However, when bidders participate in an auction, they often incur participation costs. Some costs are public information, such as the entry fees charged by the seller. However, the participation cost usually reflects the opportunity cost of the time of participating, the cost of the effort needed to learn the rules and prepare a strategy, or the cost of raising the necessary credit to participate (in spectrum auctions). These costs can be quite private since only the bidder knows his opportunity costs, ability of learning, and fundraising costs. With the existence of participation costs, not all potential bidders will participate in auctions.<sup>1</sup> In practice, usually only a small fraction of potential bidders participate in various auctions.<sup>2</sup> This has some policy implications if the seller wants to eliminate the lower tail of bidders in the distribution of willingness to pay. Bidders with low willingness to pay will stay out of the auction since they are comparing their expected revenue with their private participation costs. The remaining participants take this into account when they are bidding. Participation decisions, along with bidding strategies, should be determined simultaneously in equilibrium.

Auctions with participation costs have been investigated substantially in the literature since the salient work by Green and Laffont (1984), who study participation constraints in the Vickrey auction in a two-dimensional uniform private setting. This literature mostly focuses on second-price auctions, since the bidding behavior in second-price auctions is very straightforward. When a bidder finds it is optimal to participate in a second-price auction, he can not be better than bidding his true value. To characterize the participation equilibrium, the literature deals with the case where only one of the participation cost and the value is privately informed (Campbell 1998; Tan and Yilankaya 2006; Cao and Tian 2013). Some recent papers deal with the case in which both participation costs and value are private information (Gal et al. 2007; Cao et al. 2018), i.e., they characterize the participation equilibrium in a two-dimensional private setting.

First-price auctions with participation costs are not well-studied, since it is technically difficult to characterize the bidding functions as well as the expected revenue from participation. The difficulty hinges on that, in first-price auctions, bidding strategies are not so explicit compared to those in second-price auctions. Samuelson (1985) studies the symmetric cutoff threshold entrance equilibrium of first-price competitive procurement auctions.<sup>3</sup> Cao and Tian (2010) study the equilibria of first-price auctions when participation costs are common to all bidders. Cao et al. (2019) further study a similar problem in a two-dimensional setting where values are drawn

<sup>1</sup> Related terminology includes participation cost, participation fee, entry cost, or opportunity costs of participating in the auction. Participating costs differs from entry fee in that entry fee is part of the seller's revenue while participation cost is more general. As we only study the bidder's problem in this paper, we do not distinguish these two terminology.

<sup>2</sup> Hendricks et al. (2003) find only around 25 percent of potential bidders participate in the auctions that took place within 15 years in the united states. Bajari and Hortacsu (2003), Li and Zheng (2009), Athey S et al. (2011), Li and Zhang (2010, 2015), Krasnokutskaya and Seim (2011), Roberts and Sweeting (2013) for various auctions. See Feng, Lu and Sun (2016) for details.

<sup>3</sup> Another example is Menezes and Monteiro (2000).

from a binary distribution. This paper extends the existing literature on first price auctions with participation costs to a general two-dimensional private setting. There are at least two reasons to consider the similar questions in the First-Price-Auction setting. On the one hand, first-price auctions are frequently accommodated in practice, for instance, the auctions for tendering, particularly for government contracts and auctions for mining leases, studies of first-price auctions have received little attention, especially when both values and participation costs are privately informed. On the other hand, the bidding strategies in first-price auctions are generally more complicated, different equilibrium bidding functions generally exist due to different supports of the valuation distributions, which especially arises when bidders use different cutoffs to participate.

We present a general two-dimensional private setting on both values and participation costs in a first-price auction and characterize the equilibrium. Bidders can be heterogeneous in their types. We focus on the cutoff strategy, assuming that, all bidders participate whenever their private costs are less than or equal to some critical value, which is characterized by the expected payoff from participating, taking account of other bidders' participation decision. To characterize the equilibrium in cutoff strategies we first convert the equilibrium conditions for a profile of cutoff strategies to a system of integral equations. We then use the Schauder-Tychonoff fixed-point theorem to show that there is a solution to the system of integral equations, which establish the existence of the equilibria. We further investigate the uniqueness of the equilibrium. Particularly, when all bidders are homogeneous in their types, we show there is a unique symmetric equilibrium, i.e., all bidders use the same cutoff curves and the same bidding functions which is contingent on the number of other participants. By applying the contraction mapping theorem, in a two-bidder economy with general distributions, we show that under some mild restrictions on the joint density of values and participation costs, the equilibrium is unique. Our result shows that when the joint density is uniformly bounded, the equilibrium is unique, which typically occurs when the distribution is more dispersed.

We also compare symmetric equilibria with different distributions of participation costs. Consider two symmetric equilibria,  $A$  and  $B$ , where the distribution of participation costs in equilibrium  $A$  first-order stochastically dominates that in equilibrium  $B$ . We show that the cutoff function in equilibrium  $A$  is higher than those in equilibrium  $B$ . Intuitively, if the participation costs are more concentrated in high values, then the expected cost to participate of all bidders is higher. For each bidder  $i$ , this implies that other bidders are less likely to participate, leading to a larger expected revenue for bidder  $i$ . Hence, bidder  $i$  is willing to pay more to participate into the auction.

The main contribution of our paper is as follows. Firstly, this is the first paper to investigate the participation in first price auctions in a two-dimensional private setting, which is significant from both the theoretical and empirical point as first price auctions are used more widely in practice. Secondly, our work is technically more challenging as we have to deal with both the symmetric and even asymmetric first price auction. As such, it is technically more difficult to find the cutoffs since they are determined by the expected revenue from participating in the auction at the thresholds, which in turn depends on the more complicated bidding functions of

bidders who submit bids. Last but not the least, we provide a sufficient condition for the uniqueness of the equilibrium.

The most closely related paper is Cao et al. (2018). Our findings show that most results in second-price auctions with private participation costs still hold in first-price auctions with private participation costs. However, our results differ from theirs in at least two important aspects. First, the sufficient condition for the uniqueness of equilibrium of our paper is weaker (more general) than theirs. Second, we can compare symmetric equilibria with different distributions of participation costs if these distributions can be ranked by the first-order stochastic dominance.

The remainder of the paper is organized as follows. We describe the model in Sect. 2. In Sect. 3, we characterize the equilibrium cutoff and investigate the existence of the equilibrium. The uniqueness is addressed in Sect. 4. Concluding remarks are given in Sect. 5.

## 2 Model

Consider an independent value environment with one seller and  $n$  risk-neutral buyers (bidders). Let  $N = \{1, 2, \dots, n\}$  and  $N_{-i} = N/\{i\}$ . The seller has an indivisible object which he values at zero. The bidder with the highest bid wins the auction and pays the price equal to his bid. His payoff is equal to the difference between his valuation and the price. The other bidders have zero payoff from submitting a bid. If the highest bid is submitted by more than one bidder, there is a tie which will be broken by a fair lottery. When a bidder submits a bid, he knows who others also submit bids and thus, he can submit a contingent bid based on the identity of his competitors in the auction.

Basically, we are considering a two-stage game. In the first stage, all bidders simultaneously determine whether or not to participate in a first price auction. If a bidder chooses to submit a bid, he pays his participation cost that is not refundable, otherwise the game ends for him. In the second stage, all the bidders who pay the participation costs observe who else also participates in the auction and submit a bid. In order to submit a bid, bidder  $i$  must incur a non-refundable participation cost  $c_i$ . Bidder  $i$ 's value for the object,  $v_i$ , and his participation cost  $c_i$  are independently drawn from the distribution function  $K_i(v, c)$ , with support  $[0, 1] \times [0, 1]$ .<sup>4</sup> Let  $k_i(v, c) \geq 0$  be the corresponding density function.<sup>5</sup> Assume that  $K_i(v, c)$  is continuously differentiable, for all  $i \in N$ .

The individual action set for any bidder can be characterized as  $No \cup [0, 1]$ , where "No" denotes not submitting a bid. Bidder  $i$  incurs the participation cost  $c$  if and only if his action is different from "No". In first-price auctions, given the strategies

<sup>4</sup> The support for valuations is normalized to be  $[0, 1]$ . Bidders with participation costs higher than 1 will not participate in the auction and such bidders are of no practical interest. If the upper bounds of the supports for the participation costs are higher than 1, the above distributions on the participation costs should be interpreted as the truncated distributions of the original distributions on  $[0, 1]$ .

<sup>5</sup> We will study the special case where  $v_i$  and  $c_i$  are independently distributed in Sects. 4 and 5. When there are atoms in the distribution,  $k_i(v_i, c_i)$  can incorporate Dirac delta functions to handle the infinite density.

of all other bidders, as suggested by Lu and Sun (2007) that for any auction mechanism with participation costs, the participating and nonparticipating types of any bidder are divided by a non-decreasing and equicontinuous shutdown curve. Thus, a bidder submits a bid if and only if his expected payoff is greater than or equal to a cutoff point for costs and does not enter otherwise.

An equilibrium strategy of each bidder  $i$  is then determined by the expected payoff of participating in the auction  $c_i^*(v)$  when his value is  $v$ , together with how to bid in the first-price auction depending on the participation decision of other bidders. In equilibrium,  $c_i^*(v)$  depends on the distributions of all bidders' valuations and participation costs. We can interpret  $c_i^*(v)$  as the maximal amount bidder  $i$  would like to pay to participate in the auction when his value is  $v$ . Since participation costs are non-refundable (sunk costs), when a bidder decides to bid, his bidding strategy does not depend on his participation costs. Let  $\mathbf{c}^*(v) = \{c_1^*(v), c_2^*(v), \dots, c_n^*(v)\}$  denote the set of cutoff functions of all bidders and  $\mathbf{c}_{-i}^*(v) := \mathbf{c}^*(v) / \{c_i^*(v)\}$ .

Before describing bidding strategies, we need to introduce a few notations. Notice that when  $k$  other bidders compete with bidder  $i$ , there could be different group of bidders and the number of groups is  $C_{n-1}^k$ . Denote  $O_i^{k_l}, i \in N, k = \{0, 1, 2, \dots, n - 1\}$ , and  $l \in \{1, 2, \dots, C_{n-1}^k\}$ , as the set of the  $l$ -th group with  $k$  bidders who also participate in the auction when bidder  $i$  participates. Define  $O_i^k = \cup_{l \in \{1, 2, \dots, C_{n-1}^k\}} O_i^{k_l}$  as the set of all groups with  $k$  other participating bidders. Then, define  $O_i = \cup_{k \in \{1, 2, \dots, n-1\}} O_i^k$  as the set of all groups with at least one other participating bidder.

Let  $S_i^{k_l} = \{c_j^*(v), b_j, K_j(c_j, v_j)\}_{j \in O_i^{k_l}}, i \in N, k = \{1, 2, \dots, n - 1\}$ , and  $l \in \{1, 2, \dots, C_{n-1}^k\}$ , denote the set of the distributions of valuations and participation costs as well as cutoff functions and bidding strategies of all other bidders in the set of the  $l$ -th group with  $k$  bidders who also participate in the auction when bidder  $i$  participates. Define  $S_i^k = \cup_{l \in \{1, 2, \dots, C_{n-1}^k\}} S_i^{k_l}$  and  $S_i = \cup_{k \in \{1, 2, \dots, n-1\}} S_i^k$  accordingly. Then, the bidder  $i$ 's strategy when he faces the  $l$ -th group of other  $k$  bidders can be characterized by:

$$b_i^{k_l}(v_i, c_i, S_i^{k_l}) = \begin{cases} b_i^{k_l}(v_i, S_i^{k_l}), & \text{if } c_i \leq c_i^*(v_i), \\ No, & \text{if } c_i > c_i^*(v_i), \end{cases} \tag{1}$$

where  $b_i^{k_l}(v_i, S_i^{k_l})$  is a contingent bidding function when bidder  $i$  participates in the auction. Assume  $b_i^{k_l}(v_i, S_i^{k_l})$  is differentiable with respect to  $v_i$  for all  $i \in N$ . Note that, if bidder  $i$  enters the auction while the other bidder does not, bidder  $i$  bids zero. If the other bidder also participate, the bid of bidder  $i$  depends on 1) his own value; 2) the distributions of valuations and participation costs of other bidders; and 3) cut-off functions and bidding strategies of other bidders. In equilibrium, bidder  $i$  with

value  $v_i$  is indifferent between participating and not participating if  $c_i = c_i^*(v_i)$ .<sup>6</sup> For simplicity, we use  $b_i^{k_i}(v_i)$  to denote  $b_i^{k_i}(v_i, S_i^{k_i})$  hereafter. Define  $b_i^k(\cdot) = \{b_i^{k_i}(\cdot)\}_{i \in \{1, 2, \dots, C_{n-1}^k\}}$  and  $b_i(\cdot) = \{b_i^k(\cdot)\}_{k \in \{1, 2, \dots, n-1\}}$  accordingly. Denote  $\mathbf{b}(v) = \{b_1(v), b_2(v), \dots, b_n(v)\}$  as the set of bidding strategies of all bidders and  $\mathbf{b}_{-i}(v) := \mathbf{b}^*(v) / \{b_i(v)\}$ . Let  $v_i^{k_i}(b_i^{k_i})$  denote the corresponding inverse bidding function. Define  $v_i^k(\cdot) = \{v_i^{k_i}(\cdot)\}_{i \in \{1, 2, \dots, C_{n-1}^k\}}$  and  $v_i(\cdot) = \{v_i^k(\cdot)\}_{k \in \{1, 2, \dots, n-1\}}$  accordingly.

For the game described above, each bidder's action is to choose a cutoff for each value, i.e., a cutoff function, and decide how to bid when he participates. Thus, a (Bayesian-Nash) equilibrium of the sealed-bid first-price mechanism with participation costs is composed of bidders' cutoff strategies and participants' bidding strategies. Formally, we define the strategy profiles as follows.

**Definition 1** A strategy profile  $(\mathbf{c}^*(v), \mathbf{b}(v))$  is a (Bayesian-Nash) equilibrium of the first-price auction with private participation costs if action  $(c_i^*(v), b_i(v))$  maximizes the expected revenue of bidder  $i$ , given the strategies of other bidders  $(\mathbf{c}_{-i}^*(v), \mathbf{b}_{-i}(v))$ .

Note that, once the cutoff functions  $\mathbf{c}^*(v)$  are determined, the game is reduced to the standard first-price auction and the optimal bidding functions for participating bidders are uniquely determined (see Maskin and Riley 2003). As such, an equilibrium is fully characterized by the set of cutoffs  $\mathbf{c}^*(v)$ . Then, all the results in the paper should be interpreted in terms of cutoff functions. In general, different bidders have different cutoff functions since the distributions  $K_i(v, c)$  is bidder-specific. However, when the distributions are the same across all bidders, i.e.,  $K_i(v, c) = K(v, c)$  for all  $i \in N$ , we can define a symmetric equilibrium.

**Definition 2** If  $K_i(v, c) = K(v, c)$  for all  $i \in N$ , an equilibrium  $(\mathbf{c}^*(v), \mathbf{b}(v))$  of the first-price auction with participation costs is symmetric equilibrium if the bidders have the same cutoff function, i.e.,  $c_i^*(v) = c^*(v)$  for all  $i \in N$ .

When bidders are homogenous and use the same cutoff function, all bidders will share the same bidding function, which is contingent on the number of competitors in the auction.

**Remark 1**  $c_i^*(v_i) = 1$  implies bidder  $i$  with valuation  $v_i$  always participates and  $c_i^*(v_i) = 0$  implies bidder  $i$  with valuation  $v_i$  never participates.

<sup>6</sup> The description of the equilibria can be slightly different under different informational structures on  $K_i(v_i, c_i)$ . For example, when  $v_i$  is private information and  $c_i$  is exogenously fixed for all bidders,  $K_i(v_i, c_i) = K_i(v_i)$  (See Cao and Tian 2010), the equilibrium is described by a valuation cutoff  $v_i^*$  for bidder  $i$  such that bidder  $i$  submits a bid whenever  $v_i \geq v_i^*$ .

**Remark 2** In our case, a bidder who participates will submit bids that are contingent on the number and the composition of other participants. In certain auctions, bidders may not observe who else participates when they submit bids. In these cases, however, a bidder can only submit a uniform bid that does not depend on the number and the composition of other bidders who actually participate. The problem of each bidder in this case is similar to that in our setting. Whether changing our setting to the one where bidders cannot observe who else participates will affect the main results of this paper will be left for future research.

**Remark 3** If we ignore the participation constraints, the game would be reduced to a one-dimensional private information first-price auction. In our setting, bidders participate whenever their participation costs are sufficiently low, or put it differently, they participate whenever their value is above a threshold. Thus, the bidding strategy as a function of value for each participant will be only defined above the threshold of the value. Since the thresholds are different across participants, the bidding functions of bidders have different supports. In addition, the bidding function in the standard first-price auction has a closed form, but the bidding function in our two-dimensional setting can only be defined implicitly. Hence, the bidding functions in the two settings cannot be compared directly.

### 3 Equilibrium

Suppose, for the time being, there exists an equilibrium where bidders use the cutoff functions  $c^*(v)$  as their participation strategy. Bidder  $i$  with value  $v$  will participate in the auction and submit a bid that is contingent on information of other participating bidders  $S_i$  if and only if  $c_i \leq c_i^*(v_i)$ . Bidder  $i$  with value  $v_i$  can win the object being auctioned only when the following two cases happen. Firstly, there is no other participants. This happens with probability  $\prod_{j \in N_{-i}} \int_0^1 \int_{c_j^*(\tau)}^1 k_j(c, \tau) dc d\tau$ . Bidder  $i$  just bids zero in this case.

In the second case, there are at least one other bidder participating the auction yet all other bidders bid less than what bidder  $i$  bids. As we describe above, bidder  $i$  could face many different group of opponents. We give the following example to illustrate the simple expressions for this game. Then, we will go to the general case to show the equilibrium.

**Example 1** Let  $n = 3$ . We only characterize the cutoff function of bidder 1 and the case for other two bidders is similar. Notice that  $O_1^1 = \{2\}$  and  $O_1^2 = \{3\}$ . Hence,  $O_1^1 = \{\{2\}, \{3\}\}$ . In addition,  $O_1^2 = \{2, 3\}$ . Therefore, the expected revenue maximization problem of bidder 1 with value  $v$  is:

$$\begin{aligned}
 c_1^*(v) = & v \left( \int_0^1 \int_{c_2^*(\tau)}^1 k_2(c, \tau) dc d\tau \right) \left( \int_0^1 \int_{c_3^*(\tau)}^1 k_3(c, \tau) dc d\tau \right) \\
 & + \max_{b_1^{l_1}} (v - b_1^{l_1}) \left( \int_0^{v_2^1(b_1^{l_1})} \int_0^{c_2^*(\tau)} k_2(c, \tau) dc d\tau \right) \left( \int_0^1 \int_{c_3^*(\tau)}^1 k_3(c, \tau) dc d\tau \right) \\
 & + \max_{b_1^{l_2}} (v - b_1^{l_2}) \left( \int_0^{v_3^1(b_1^{l_2})} \int_0^{c_3^*(\tau)} k_3(c, \tau) dc d\tau \right) \left( \int_0^1 \int_{c_2^*(\tau)}^1 k_2(c, \tau) dc d\tau \right) \\
 & + \max_{b_1^{l_3}} (v - b_1^{l_3}) \left( \int_0^{v_2^2(b_1^{l_3})} \int_0^{c_2^*(\tau)} k_2(c, \tau) dc d\tau \right) \left( \int_0^{v_3^2(b_1^{l_3})} \int_0^{c_3^*(\tau)} k_3(c, \tau) dc d\tau \right).
 \end{aligned}$$

The first line is the payoff when all other bidders do not participate. The second and third line are the payoff when bidder 3 and 2 does not participate and bidder 2 and 3 participates but bids less than what bidder 1 does. The last line is the payoff when both other bidders participate but bid less. By the envelope theorem, we have

$$\begin{aligned}
 c_1'(v) = & \left( \int_0^1 \int_{c_2^*(\tau)}^1 k_2(c, \tau) dc d\tau \right) \left( \int_0^1 \int_{c_3^*(\tau)}^1 k_3(c, \tau) dc d\tau \right) \\
 & + \left( \int_0^{v_2^1(b_1^{l_1}(v))} \int_0^{c_2^*(\tau)} k_2(c, \tau) dc d\tau \right) \left( \int_0^1 \int_{c_3^*(\tau)}^1 k_3(c, \tau) dc d\tau \right) \\
 & + \left( \int_0^{v_3^1(b_1^{l_2}(v))} \int_0^{c_3^*(\tau)} k_3(c, \tau) dc d\tau \right) \left( \int_0^1 \int_{c_2^*(\tau)}^1 k_2(c, \tau) dc d\tau \right) \\
 & + \left( \int_0^{v_2^2(b_1^{l_3}(v))} \int_0^{c_2^*(\tau)} k_2(c, \tau) dc d\tau \right) \left( \int_0^{v_3^2(b_1^{l_3}(v))} \int_0^{c_3^*(\tau)} k_3(c, \tau) dc d\tau \right).
 \end{aligned}$$

Based on the above example, now we consider the case where the  $l$ -th group of  $k$  other bidders participate yet all these  $k$  bidders bid less than  $b_i^{k_l}$ . In this case, bidder  $i$  with value  $v$  bids  $b_i$  to maximize his expected revenue:

$$\max_{b_i} (v - b_i) P(b_i; O_i^{k_l}), \tag{2}$$

where

$$P(b_i; O_i^{k_l}) := \prod_{j \in O_i^{k_l}} \int_0^{v_j^{k_l}(b_i)} \int_0^{c_j^*(\tau)} k_j(c, \tau) dc d\tau.$$

The first part is the revenue conditional on winning. The second part is the probability of facing the  $l$ -th group of  $k$  other bidders who all bid less than  $b$ . To ensure these probabilities are well-defined, we assume



$$\prod_{j \in O_i^{k_l}} \int_0^{v_j^{k_l}(b_i)} \int_0^{c_j^*(\tau)} k_j(c, \tau) dc d\tau = 1,$$

whenever  $O_i^{k_l}$  are empty sets, respectively, for all  $i \in N$ ,  $k = \{0, 1, 2, \dots, n - 1\}$ , and  $l \in \left\{ 1, 2, \dots, \frac{(n-1)!}{k!(n-1-k)!} \right\}$ .

Notice that the bidding strategy  $b_i^{k_l}(v)$  solves this problem. Maskin and Riley (2003) show that there is a unique bidding equilibrium for the above first-price auction. Denote the maximized expected revenue of bidder  $i$  when he competes with the  $l$ -th group of  $k$  other bidders by:

$$R_i^{k_l}(v) = \left( v - b_i^{k_l}(v) \right) P\left( b_i^{k_l}; O_i^{k_l} \right) Q\left( N_{-i} / O_i^{k_l} \right), \tag{3}$$

where

$$Q\left( N_{-i} / O_i^{k_l} \right) := \prod_{m \in N_{-i} / O_i^{k_l}} \int_0^1 \int_{c_m^*(\tau)}^1 k_m(c, \tau) dc d\tau,$$

is the probability of all other bidders do not participate. To ensure these probabilities are well-defined, we assume

$$\prod_{j \in O_i^{k_l}} \int_0^{v_j^{k_l}(b_i)} \int_0^{c_j^*(\tau)} k_j(c, \tau) dc d\tau = 1, \quad \prod_{m \in N_{-i} / O_i^{k_l}} \int_0^1 \int_{c_m^*(\tau)}^1 k_m(c, \tau) dc d\tau = 1,$$

whenever  $N_{-i} / O_i^{k_l}$  are empty sets, respectively, for all  $i \in N$ ,  $k = \{0, 1, 2, \dots, n - 1\}$ , and  $l \in \left\{ 1, 2, \dots, \frac{(n-1)!}{k!(n-1-k)!} \right\}$ . Hence, the total maximized expected revenue of bidder  $i$  with value  $v$  when there is at least one opponent is given by:

$$R_i(v) = \sum_{k=1}^{n-1} \sum_{l=1}^{\frac{(n-1)!}{k!(n-1-k)!}} R_i^{k_l}(v). \tag{4}$$

Therefore, in equilibrium, the following condition must hold:

$$c_i^*(v) = v \left( \prod_{j \in N_{-i}} \int_0^1 \int_{c_j^*(\tau)}^1 k_j(c, \tau) dc d\tau \right) + R_i(v), \tag{5}$$

for all  $i \in N$ . For the sake of simplicity, we define  $\Phi_i(v)$  as follows:

$$\Phi_i(s) := \prod_{j \in N_{-i}} \int_0^1 \int_{c_j^*(\tau)}^1 k_j(c, \tau) dc d\tau + \sum_{k=1}^{n-1} \sum_{l=1}^{\frac{(n-1)!}{k!(n-1-k)!}} P\left( b_i^{k_l}(s); O_i^{k_l} \right) Q\left( N_{-i} / O_i^{k_l} \right).$$

Notice that  $\Phi_i(v)$  is the probability of winning for bidder  $i$  with value  $v$ . Notice that  $O_i^0$  is empty, we have  $N_{-i} = N_{-i}/O_i^0$  and, for all  $i \in N$ ,

$$\begin{aligned} & \left( \prod_{j \in O_i^0} \int_0^{v_j^{k_j'}(b_i^{k_l}(v))} \int_0^{c_j^*(\tau)} k_j(c, \tau) dc d\tau \right) \left( \prod_{j \in N_{-i}/O_i^{k_l}} \int_0^1 \int_{c_j^*(\tau)}^1 k_j(c, \tau) dc d\tau \right) \\ &= \prod_{j \in N_{-i}} \int_0^1 \int_{c_j^*(\tau)}^1 k_j(c, \tau) dc d\tau. \end{aligned}$$

Hence, we can rewrite  $\Phi_i(v)$  as follows:

$$\Phi_i(v) = \sum_{k=0}^{n-1} \sum_{l=1}^{\frac{(n-1)!}{k!(n-1-k)!}} P(b_i^{k_l}(v); O_i^{k_l}) Q(N_{-i}/O_i^{k_l}),$$

for all  $i \in N$ .

Next, let  $\bar{P}(b_i^{k_l}(v); O_i^{k_l})$  denote the probability of losing for bidder  $i$  in  $O_i^{k_l}$  when his bid is  $b_i^{k_l}(v)$ , for  $i \in N$ ,  $k = \{1, 2, \dots, n-1\}$ , and  $l \in \left\{1, 2, \dots, \frac{(n-1)!}{k!(n-1-k)!}\right\}$ . For each  $O_i^{k_l}$ , rank bidders by their valuations in ascending order. In particular, if the valuation of a bidder in  $O_i^{k_l}$  is the  $p$ -th largest, the index of this bidder is  $p \in \{1, 2, \dots, k\}$ . Define the following two auxiliary probabilities:

$$\begin{aligned} \underline{Q}_p^{k_l} &= \begin{cases} \prod_{j=1}^{p-1} \int_0^{v_j^{k_j'}(b_i^{k_l}(v))} \int_0^{c_j^*(\tau)} k_j(c, \tau) dc d\tau, & \text{if } 2 \leq p \leq k, \\ 1, & \text{if } p = 1 \end{cases} \\ \bar{Q}_p^{k_l} &= \begin{cases} \prod_{j=p+1}^k \int_0^1 \int_0^{c_j^*(\tau)} k_j(c, \tau) dc d\tau, & \text{if } 1 \leq p \leq k-1, \\ 1, & \text{if } p = k \end{cases}. \end{aligned}$$

By the definition of  $\bar{P}(b_i^{k_l}(v); O_i^{k_l})$ , we have:

$$\bar{P}(b_i^{k_l}(v); O_i^{k_l}) := \sum_{p=1}^k \left( \left( \int_{v_p^{k_p'}(b_i^{k_l}(v))}^1 \int_0^{c_p^*(\tau)} k_p(c, \tau) dc d\tau \right) \underline{Q}_p^{k_l} \bar{Q}_p^{k_l} \right).$$

Now, since  $\Phi_i(\cdot)$  is the probability of winning for bidder  $i$ , we have:

$$\Phi_i(v) = 1 - \sum_{k=0}^{n-1} \sum_{l=1}^{\frac{(n-1)!}{k!(n-1-k)!}} \left( \bar{P}(b_i^{k_l}(v); O_i^{k_l}) Q(N_{-i}/O_i^{k_l}) \right). \tag{6}$$

Proposition 1 solves the expected revenue maximization problem and derives the equilibrium condition that the cutoff function has to satisfy.

**Proposition 1** *Suppose there exists an equilibrium cutoff  $c^*(v)$  and all bidding functions are differentiable. Then,  $c^*(v)$  is characterized by:*

$$c_i^*(v) = \int_0^v \Phi_i(s) ds, \tag{7}$$

with  $c_i^*(0) = 0$  for all  $i \in N$ .  $c_i^*(v)$  is differentiable with

$$c_i^{*'}(v) = \Phi_i(v), \tag{8}$$

for all  $i \in N$ .  $c_i^{*'}(v)$  is also differentiable.

**Proof** See the [Appendix](#). □

Notice that the first part of  $c_i^{*'}(v)$  is the probability that none of the other  $n - 1$  bidders participates and the second part is the probability that at least one of the other  $n - 1$  participating bidders who all bid less than  $b_i$ . Hence,  $c_i^{*'}(v)$ , as the sum of the two parts, is the probability of winning for bidder  $i$ . We illustrate the above analysis in the three-bidder example below.

**Remark 4**  $v_j(b_i(v))$  gives the value of bidder  $j$  such that he bids exactly the same as bidder  $i$  with value  $v$ . When all bidders use the same bidding function,  $v_j(b_i(v)) = v$ . This happens in second price auctions or in a symmetric equilibrium in first price auctions.

Before establishing the existence and uniqueness of the equilibrium, we first characterize some useful properties of the equilibrium cutoff function  $c^*(v)$ . Proposition 2 summarizes these properties that will be used in the proof of Theorem 1.

**Proposition 2** *Suppose there exists an equilibrium cutoff  $c^*(v)$ . It has the following properties: for  $i \in N$ ,*

- (i)  $c_i^{*'}(v) \geq 0$  and  $c_i^{*''}(v) \geq 0$ ;
- (ii)  $0 \leq c_i^*(v) \leq v$ ;
- (iii)  $c_i^{*'}(1) = 1$ .

**Proof** See the [Appendix](#). □

The first part of Proposition 2 states that the expected payoff from participating of a bidder is increasing and convex in his value  $v$ . The second part shows that a bidder will not be willing to pay more than his value to participate in the auction. The last part finds that the marginal willingness to pay to participate in the auction of the bidder with value  $v = 1$  is also 1. The intuition is that when his value for the object is 1, he will almost surely win the object, and the marginal willingness to pay is equal to his value which is 1.

We establish our main results, which are the existence and uniqueness of the symmetric equilibrium, in Theorem 1.

**Theorem 1** *There always exists an equilibrium cutoff function  $c^*(v)$  such that all bidders participate whenever  $c_i \leq c_i^*(v)$  and Eq. (7) holds for all  $i \in N$  simultaneously.*

**Proof** See the [Appendix](#). □

**Remark 5** In the proof, we construct a mapping of  $c^*(v)$  from a space to itself. In the [Appendix](#), we show that this space is a compact convex nonempty subset of a locally convex topological space and the mapping is continuous. Then we establish the existence of equilibrium using the Schauder-Tychonoff fixed-point theorem, which states that any continuous mapping from a nonempty compact convex subset of a locally convex topological space to itself has a fixed point

## 4 Uniqueness

In this section, we investigate the uniqueness of equilibrium in two interesting yet important settings. On the one hand, we establish the uniqueness of equilibrium in the case where all bidders are ex ante homogeneous. For any two symmetric equilibria where valuations and participation costs are independently distributed, we show that the equilibrium cutoff is lower when the distribution of participation costs is first-order stochastically dominated. On the other hand, we show there is a unique equilibrium in the case where the distribution functions are heterogeneous across bidders.

### 4.1 Symmetric equilibrium

We first establish uniqueness of symmetric equilibrium. Assume all bidders are homogenous with a common joint distribution of valuations and participation costs.  $K_i(\cdot, \cdot) = K(\cdot, \cdot)$  and  $k_i(\cdot, \cdot) = k(\cdot, \cdot)$  for all  $i \in N$ . We focus on the symmetric equilibrium in which all bidders share the same cutoff function  $c_i^*(v) = c^*(v)$  and the same bidding function  $b_i(v) = b(v)$ , for all  $i \in N$ . In this case, once a bidder participates, his bid depends only on the number of other participating bidders. Let  $b^k(v)$  be the bidding function of a bidder with value  $v$  when there are  $k$  other participants. Then, the cutoff function satisfies:

$$c^*(v) = v \left[ \int_0^1 \int_{c^*(\tau)}^1 k(c, \tau) dc d\tau \right]^{n-1} + \sum_{k=1}^{n-1} C_{n-1}^k \left( \max_{b^k} (v - b^k) \left( \int_0^{v^k(b^k)} \int_0^{c^*(\tau)} k(c, \tau) dc d\tau \right)^k \left( \int_0^1 \int_{c_j^*(\tau)}^1 k(c, \tau) dc d\tau \right)^{n-1-k} \right).$$

**Proposition 3** characterizes the cutoff  $c^*(v)$  in an symmetric equilibrium.

**Proposition 3** *When all bidders are homogenous, the symmetric equilibrium is characterized by*

$$c^*(v) = \int_0^v \left[ 1 - \int_t^1 \int_0^{c^*(\tau)} k(c, \tau) dc d\tau \right]^{n-1} dt, \tag{9}$$

with  $c^*(v) \in [0, v]$  and  $c^*(0) = 0$ .  $c^*(v)$  is differentiable with

$$c^{*'}(v) = \left( 1 - \int_v^1 \int_0^{c^*(\tau)} k(c, \tau) dc d\tau \right)^{n-1} \geq 0,$$

and  $c^{*'}(1) = 1$ .  $c^{*'}(v)$  is also differentiable with  $c^{*''}(v) \geq 0$ .

**Proof** See the [Appendix](#). □

Equation (9) in Proposition 3 is the functional equation that determines the equilibrium cutoff function  $c^*(v)$ . The functional Eq. (9) and its derivative in the expression right below (9) are derived from the equation right above Proposition 3. The left-hand side of this equation is the cutoff value of the participation cost for a bidder with value  $v$ . The right-hand side of this equation is the expected revenue of a bidder with value  $v$  from participating. In particular, the first term is the revenue of the bidder when there is no other participant and the second term is the summation of the expected revenue of the bidder in all other cases. We illustrate the symmetric equilibrium in the three-bidder example below.

**Example 2** Assume all bidders are homogenous:  $K_i(\cdot, \cdot) = K(\cdot, \cdot)$ . The expected revenue maximization problem of any bidder with value  $v$  is:

$$\begin{aligned} c^*(v) = & v \left( \int_0^1 \int_{c^*(\tau)}^1 k(c, \tau) dc d\tau \right)^2 \\ & + 2 \max_{b^1} (v - b^1) \left( \int_0^{v^1(b^1)} \int_0^{c^*(\tau)} k(c, \tau) dc d\tau \right) \left( \int_0^1 \int_{c^*(\tau)}^1 k(c, \tau) dc d\tau \right) \\ & + \max_{b^2} (v - b^2) \left( \int_0^{v^2(b^2)} \int_0^{c^*(\tau)} k(c, \tau) dc d\tau \right)^2. \end{aligned}$$

The first line is the payoff when all other bidders do not participate. The second line is the payoff when only one other bidder participates but bids less than what bidder 1 does. The last line is the payoff when both other bidders participate but bid less. By the envelope theorem, we have

$$\begin{aligned}
c^{*f}(v) &= \left( \int_0^1 \int_{c^*(\tau)}^1 k(c, \tau) dc d\tau \right)^2 \\
&\quad + 2 \left( \int_0^v \int_0^{c^*(\tau)} k(c, \tau) dc d\tau \right) \left( \int_0^1 \int_{c^*(\tau)}^1 k(c, \tau) dc d\tau \right) \\
&\quad + \left( \int_0^v \int_0^{c^*(\tau)} k(c, \tau) dc d\tau \right)^2 \\
&= \left( 1 - \int_v^1 \int_0^{c^*(\tau)} k(c, \tau) dc d\tau \right)^2.
\end{aligned}$$

We establish the uniqueness of symmetric equilibrium in Theorem 2.

**Theorem 2** (Uniqueness of Symmetric Equilibrium) *Suppose that all bidders have the same distribution function  $K(\cdot, \cdot)$ . There is a unique symmetric equilibrium where all bidders use the identical cutoff strategy  $c^*(\cdot)$ .*

**Proof** See the [Appendix](#). □

**Remark 6** In the literature, Cao and Tian (2010) and Cao et al. (2019) are the very few papers that also establish the existence and uniqueness of equilibrium in first-price auctions with participation costs. However, Cao and Tian (2010) assume that participation costs are common knowledge while the private value follows a binary distribution in Cao et al. (2019). Thus, to some extent, these authors investigate special cases of this paper.

**Corollary 1**  $c^*(v)$  decreases as  $n$  increases.

**Proof** See the [Appendix](#). □

Intuitively as there are more potential bidders, the expected payoff from participating will be less as bidders face more competition and thus bidder are less willing to participate.

Next, we investigate how the unique symmetric equilibrium cutoff function responds to changes of the distribution of participation costs. Note that Proposition 3 implies that the cutoff function of a symmetric equilibrium is determined in Eq. (9) solely by the joint distributions of valuations and participation costs. It is interesting to compare equilibrium cutoff functions that corresponds to different distributions of participation costs. To this end, we assume valuations and participation costs are independently distributed, i.e.  $K(v, c) = F(v)G(c)$  and  $k(v, c) = f(v)g(c)$  with  $f(v) = F'(v)$  and  $g(c) = G'(c)$ . We compare equilibrium cutoff functions of symmetric equilibria, where all bidders share an equilibrium cutoff function  $c^*(\cdot)$ .

More specifically, consider two sets of distributions  $A$  and  $B$ , where  $F_A(v) = F_B(v)$  and  $G_A(c)$  first-order stochastically dominates  $G_B(c)$ , i.e.  $G_A < G_B$ .<sup>7</sup> We rewrite Eq. (9) for  $q \in \{A, B\}$  as follows:

$$c_q^*(v) = \int_0^v \left[ 1 - \int_t^1 f_q(\tau) G_q(c_q^*(\tau)) d\tau \right]^{n-1} dt.$$

Theorems 1 and 2 ensure that there is a unique equilibrium for distribution  $q \in \{A, B\}$ . Proposition 4 presents the comparative statics of the symmetric equilibrium.

**Proposition 4**  $c_A^*(v) > c_B^*(v)$  for all  $v \in (0, 1]$  if  $G_A$  first-order stochastically dominates  $G_B$ , where  $c_q^*(v)$  is the equilibrium cutoff function for distribution  $q \in \{A, B\}$ .

*Proof* See the Appendix. □

**Remark 7** Intuitively, if the participation costs are more concentrated in high values, then the expected cost to participate of all bidders is higher. For each bidder  $i$ , this implies that other bidders are less likely to participate. Hence, the expected revenue for bidder  $i$  is larger. This in turn means bidder  $i$  is willing to pay more to participate into the auction. Notice also that even though the equilibrium cutoff is higher, the probability of participation for each bidder declines.

### 4.2 Uniqueness of equilibrium

Uniqueness of the symmetric equilibrium does not imply the uniqueness of equilibrium. Even when bidders are homogenous, they may still participate in the auction asymmetrically, i.e., they use different cutoff curves. In the next section, under a general distribution, we provide a sufficient condition for the uniqueness of equilibrium.

In this subsection, we consider the case where there are two bidders. In this case, Eqs. (7) and (8) are reduced to:

$$\begin{aligned} c_i^*(v) &= \int_0^v \left[ 1 - \int_{v_j(b_i(s))}^1 \int_0^{c_j^*(\tau)} k_j(c, \tau) dc d\tau \right] ds, \\ c_i^{*j}(v) &= 1 - \int_{v_j(b_i(v))}^1 \int_0^{c_j^*(\tau)} k_j(c, \tau) dc d\tau, \end{aligned} \tag{10}$$

where  $i, j \in \{1, 2\}$  and  $i \neq j$ . To show the uniqueness of asymmetric equilibrium, we rely on the contraction mapping theorem. Proposition 5 presents our result.

<sup>7</sup> Subscripts  $A$  and  $B$  are used to indicate variables associated with different distributions.

**Proposition 5** (Uniqueness of Equilibrium) *Suppose  $n = 2$ . If, for  $i, j \in \{1, 2\}$  and  $j \neq i$ , (i)  $K_i(c_i, v_i)$  is differentiable; and (ii)  $\sup_{c, v \in [0, 1]} k_i(c, v) < \frac{1}{\int_0^1 [1 - v_i(b_j(v))] dv}$ , then there exists a unique equilibrium.*

**Proof** See the [Appendix](#). □

**Corollary 2** *Suppose  $n = 2$ . If, for  $i, j \in \{1, 2\}$  and  $j \neq i$ , (i)  $K_i(c_i, v_i)$  is differentiable; and (ii) the valuations and participation costs are independently distributed, i.e.  $K_i(v_i, c_i) = F_i(v_i)G_i(c_i)$ ; and (iii)  $\sup_{c \in [0, 1]} g_i(c) < \frac{1}{\int_0^1 [1 - F_i(v_i(b_j(v)))] dv}$ , then there exists a unique equilibrium. Condition (iii) holds if condition (ii) in Proposition 5 is satisfied.*

**Proof** See the [Appendix](#). □

**Remark 8** A few remarks on the uniqueness of equilibrium for the two-bidder setting are in order.

- (1) The sufficient conditions in Proposition 5 can be easily satisfied. For instance, when participation costs are jointly uniformly distributed on  $[0, 1] \times [0, 1]$  for both bidders, the supremum of the density for participation costs is 1, i.e.,  $\sup_{c, v \in [0, 1]} k_i(c, v) = 1$ , and clearly  $\int_0^1 [1 - v_i(b_j(v))] dv < 1$  for any  $K_i(\cdot, \cdot)$  on  $[0, 1] \times [0, 1]$ . Thus, in this case, the equilibrium is unique regardless of the distributions of valuations and participation costs.
- (2) It can be concluded that in the two-bidder economy, when the bidders are homogeneous and when the sufficient conditions in Proposition 5 are satisfied, the symmetric equilibrium is the unique equilibrium.
- (3) In the proof of the uniqueness of equilibrium, we make use of the mean value theorem before using the contraction mapping theorem. For  $n \geq 3$ , there are more product terms in the expected revenue of each bidder, which makes the application of the mean value theorem less tractable.
- (4) To understand the sufficient condition in Proposition 5, notice that our idea to show the uniqueness of equilibrium is to use the contraction mapping theorem. The sufficient condition ensures that the mapping defined in Eq. (10) is indeed a contraction mapping. Intuitively, this sufficient condition requires that the distribution of  $c_i$  and  $v_i$  are dispersed enough, or  $v_i$  is more concentrated on low values.

## 5 Concluding remarks

We study the existence and uniqueness of equilibrium in first-price auctions when bidders' values and participation costs are both private information. We show that under general distribution functions, there always exists an equilibrium in which each bidder uses a cutoff strategy. When bidders are ex ante homogeneous, there is a unique symmetric equilibrium. When there are two heterogeneous bidders, we



provide a mild sufficient condition for the uniqueness of the equilibrium. Future research could focus on identifying sufficient conditions that are easier to verify to guarantee uniqueness of equilibrium in general environments. Beside that, this paper only focus on the equilibrium analysis of the bidders. The next step would be to study the implications of the private participation costs on the seller's expected revenue as well as the social welfare. We would leave it for a future work.

## Appendix

### Proof of Proposition 1

First, notice that it follows from the inverse function theorem that all inverse bidding functions are also differentiable. Hence,  $\Phi_i(v)$  and thus  $R_i(v)$  are differentiable for all  $i \in N$ . It follows that  $c_i^*(v)$  is differentiable for all  $i \in N$ . Then, we can apply the envelope theorem in Eq. (5). Differentiating Eq. (5) derives:

$$c_i^{*'}(v) = \Phi_i(v),$$

for all  $i \in N$ . Since  $\Phi_i(v)$  is differentiable,  $c_i^{*'}(v)$  is also differentiable. Last, integrating Eq. (8) from 0 to 1 and noticing  $c_i^*(0) = 0$  for all  $i \in N$ , we derive Eq. (7). To see why  $c_i^*(0) = 0$ , notice that since  $0 \leq b_i^{k_l}(v) \leq 0$ , we have  $b_i^{k_l}(0) = 0$ , for all  $i \in N, k = \{1, 2, \dots, n - 1\}$ , and  $l \in \{1, 2, \dots, C_{n-1}^k\}$ . It follows that  $\Phi_i(0) = 0$ , for all  $i \in N$ . Hence, evaluating Eq. (5) at  $v = 0$  yields  $c_i^*(0) = 0$ , for all  $i \in N$ . This completes the proof.

### Proof of Proposition 2

First, (i) Eq. (8) implies  $c_i^{*'}(v) \geq 0$  since  $K_i(\cdot, \cdot)$  is a distribution function defined on  $[0, 1] \times [0, 1]$ . Since  $c_i^{*'}(v)$  is differentiable for all  $i \in N$ , it is clear that Eq. (8) implies  $c_i^{*'}(v)$  is increasing:

$$c_i^{*''}(v) = \Phi_i'(v) \geq 0,$$

for all  $i \in N$ . Second, Eq. (7) implies  $c_i^*(v) \geq 0$ . Notice that  $\Phi_i(\cdot) \in [0, 1]$  since it is the probability of winning for bidder  $i$ . Equation (7) implies:

$$c_i^*(v) = \int_0^v \Phi_i(s) ds \leq \int_0^v ds = v.$$

Last, notice that:

$$c_i^{*'}(v) = \Phi_i(v) = 1 - \sum_{k=0}^{n-1} \sum_{l=1}^{C_{n-1}^k} \left( \bar{P}(b_i^{k_l}(v); O_i^{k_l}) Q(N_{-i}/O_i^{k_l}) \right),$$

where

$$\bar{P}(b_i^{k_l}(v); O_i^{k_l}) := \sum_{p=1}^k \left( \left( \int_{v_p^{k_l'}}(b_i^{k_l}(v)) \int_0^{c_p^{*'}(\tau)} k_p(c, \tau) dc d\tau \right) \frac{Q^{k_l} \bar{Q}_p^{k_l}}{\underline{Q}_p} \right).$$

Notice that Maskin and Riley (2003) show that if the upper endpoint of the support of the valuation distributions is the same for all bidders in first-price auctions, then the upper endpoints of the supports of all bidders' equilibrium bid distributions are the same. Since  $v_i$  have the same upper endpoint of their distributions, we have  $b_i(1) = b^*$  and  $v_j(b_i(1)) = 1$ , where  $j \neq i$  and  $b^*$  is a constant. Evaluating  $c_i^{*'}(v)$  at  $v = 1$  derives  $c_i^{*'}(1) = 1$  since  $v_j^{k_l'}(b_i^{k_l}(1)) = 1$  for all  $i \in N$ . This completes the proof.

**Proof of Theorem 1**

We establish the existence by applying the Schauder-Tychonoff fixed-point theorem, which states that any continuous mapping from a compact convex non-empty subset of a locally convex topological space to itself has a fixed point (Burton 2005). For all  $i \in N$ , define a mapping  $h_i : [0, 1] \rightarrow [0, 1]$  as follows:

$$h_i(t_i, \mathbf{c}^*(\cdot)) := \Phi(t_i; \mathbf{c}^*(\cdot)).$$

Since  $K(\cdot, \cdot)$  is integrable over both arguments and bidding functions are differentiable (hence continuous),  $h_i(t_i, \mathbf{c}^*(\cdot))$  is a continuous mapping from  $[0, 1] \times [0, 1]^n \rightarrow [0, 1]$ , for all  $i \in N$ . Define  $H(t, \mathbf{c}^*(\cdot)) : [0, 1]^n \times [0, 1]^n \rightarrow [0, 1]^n$  as follows:

$$H(t, \mathbf{c}^*(\cdot)) = (h_1(t_1, \mathbf{c}^*(\cdot)), h_2(t_2, \mathbf{c}^*(\cdot)), \dots, h_n(t_n, \mathbf{c}^*(\cdot)))'$$

Since  $h_i(t_i, \mathbf{c}^*(\cdot))$  is continuous for all  $i \in N$ ,  $H(t, \mathbf{c}^*(\cdot))$  is also a continuous mapping. Next, let  $C([0, 1])$  be the space of continuous functions  $\psi$  mapping from  $[0, 1]$  to  $[0, 1]^n$  with the sup norm on  $[0, 1]$ :

$$\|\psi\| = \sup_{v \in [0,1]} |\psi(v)|,$$

where  $|\cdot|$  is a norm in  $R^n$  and for any  $x \in R^n$ ,  $|\cdot|$  is defined as  $|x| = \max_{i \in N} |x_i|$ . Define

$$M = \{\psi \in C([0, 1]) : \|\psi\| \leq 1\}.$$

Notice that  $c^*(\cdot) \in M$ . By definition,  $M$  is equicontinuous and equibounded. It follows from the Arzela-Ascoli theorem (Burton 2005) that  $M$  is compact. In addition,  $M$  is certainly convex by definition.

Define a mapping  $T : M \rightarrow M$  as follows:

$$(Tc^*)(v) := \int_0^v H(t, c^*(\cdot))dt.$$

We next establish that  $T$  is a continuous mapping. To this end, let  $\phi, \varphi \in M$ . It suffices to show that for any  $\epsilon > 0$ , there is a  $\delta$  such that  $\|\phi - \varphi\| < \delta$  implies  $\|T\phi - T\varphi\| < \epsilon$ . Since  $h_i(t_i, c^*(\cdot))$  is a continuous mapping from  $[0, 1] \times [0, 1]^n \rightarrow [0, 1]$ , for any  $\epsilon > 0$ , and  $t_i \in [0, 1]^n$ , there exists an  $\delta > 0$  such that  $|h_i(t_i, \phi(\cdot)) - h_i(t_i, \varphi(\cdot))| < \epsilon$  when  $\|\phi - \varphi\| < \delta$ , for all  $i \in N$ . To establish the continuity of  $T$ , notice that for  $\delta$  such that  $\|\phi - \varphi\| < \delta$ , we have:

$$\begin{aligned} \|T\phi - T\varphi\| &= \sup_{v \in [0,1]} \left| \int_0^v H(t, \phi(\cdot))dt - \int_0^v H(t, \varphi(\cdot))dt \right| \\ &= \sup_{v \in [0,1]} \left( \max_{i \in N} \left| \int_0^v h_i(t_i, \phi(\cdot))dt_i - \int_0^v h_i(t_i, \varphi(\cdot))dt_i \right| \right) \\ &\leq \sup_{v \in [0,1]} \left( \max_{i \in N} \int_0^v |h_i(t_i, \phi(s)) - h_i(t_i, \varphi(s))| dt_i \right) \\ &< \sup_{v \in [0,1]} \left( \max_{i \in N} \int_0^v \epsilon dt_i \right) \\ &= \sup_{v \in [0,1]} (v\epsilon) = \epsilon. \end{aligned}$$

It follows from the Schauder-Tychonoff fixed-point theorem that  $T$  has a fixed point in  $M$ . Hence, there exists at least an equilibrium cutoff  $c^*(\cdot)$ . This completes the proof.

**Proof of Proposition 3**

First, notice that  $c^*(v)$  is differentiable. Then, we can applying the envelope theorem in Eq. (5). Notice that  $v^k(b^k(v)) = v$  in a symmetric equilibrium. Differentiating Eq. (5) derives:

$$\begin{aligned}
c_i^{*/j}(v) &= \left[ \int_0^1 \int_{c^*(\tau)}^1 k(c, \tau) dc d\tau \right]^{n-1} \\
&+ \sum_{k=1}^{n-1} \left( C_{n-1}^k \left( \int_0^v \int_0^{c^*(\tau)} k(c, \tau) dc d\tau \right)^k \left( \int_0^1 \int_{c^*(\tau)}^1 k(c, \tau) dc d\tau \right)^{n-1-k} \right), \\
&= \sum_{k=0}^{n-1} \left( C_{n-1}^k \left( \int_0^v \int_0^{c^*(\tau)} k(c, \tau) dc d\tau \right)^k \left( \int_0^1 \int_{c^*(\tau)}^1 k(c, \tau) dc d\tau \right)^{n-1-k} \right) \\
&= \left( \int_0^v \int_0^{c^*(\tau)} k(c, \tau) dc d\tau + \int_0^1 \int_{c^*(\tau)}^1 k(c, \tau) dc d\tau \right)^{n-1} \\
&= \left( \int_0^v \int_0^{c^*(\tau)} k(c, \tau) dc d\tau + \int_v^1 \int_{c^*(\tau)}^1 k(c, \tau) dc d\tau + \int_0^v \int_{c^*(\tau)}^1 k(c, \tau) dc d\tau \right)^{n-1} \\
&= \left( \int_0^v \int_0^1 k(c, \tau) dc d\tau + \int_v^1 \int_{c^*(\tau)}^1 k(c, \tau) dc d\tau \right)^{n-1} \\
&= \left( 1 - \int_v^1 \int_0^1 k(c, \tau) dc d\tau + \int_v^1 \int_{c^*(\tau)}^1 k(c, \tau) dc d\tau \right)^{n-1} \\
&= \left( 1 - \int_v^1 \int_0^{c^*(\tau)} k(c, \tau) dc d\tau \right)^{n-1}.
\end{aligned}$$

It follows that  $c_i^{*/j}(v)$  is also differentiable and  $c_i^{*/j}(1) = 1$ . Last, integrating Eq. (8) from 0 to 1 and noticing  $c_i^{*/j}(0) = 0$  for all  $i \in N$ , we derive Eq. (7). To see why  $c_i^{*/j}(0) = 0$ , notice that  $b^k(0) = 0$  and  $v^k(0) = 0$ , since  $v^k(\cdot)$  is the inverse bidding function. It follows that  $c_i^{*/j}(0) = 0$ . This completes the proof.

## Proof of Theorem 2

The existence of the symmetric equilibrium can be established by the Schauder-Tychonoff fixed-point theorem. Here we only need to prove the uniqueness of the symmetric equilibrium. Suppose, by contradiction, that we have two different symmetric equilibria  $x(v)$  and  $y(v)$ . Then we have:

$$\begin{aligned}
x'(v) &= \left[ 1 - \int_v^1 \int_0^{x(\tau)} k(\tau, c) dc d\tau \right]^{n-1}, \\
y'(v) &= \left[ 1 - \int_v^1 \int_0^{y(\tau)} k(\tau, c) dc d\tau \right]^{n-1}.
\end{aligned}$$

Without loss of generality, suppose  $x(1) > y(1)$ , then by the continuity of  $x(v)$  and  $y(v)$ , there exists a  $v^* \in [0, 1)$  such that  $x(v^*) = y(v^*) := c(v^*)$  and  $x(\tau) > y(\tau)$  for all  $\tau \in (v^*, 1]$  by noting that  $x(0) = y(0)$ . Consider two mutually exclusive cases. First, if  $k(v, c) > 0$  with positive probability measure on  $(v^*, 1) \times (c(v^*), 1)$ , then  $x(\tau) > y(\tau)$  for all  $\tau \in (v^*, 1]$  implies that

$$\int_0^{x(\tau)} k(\tau, c)dc > \int_0^{y(\tau)} k(\tau, c)dc,$$

for all  $\tau \in (v^*, 1)$ . It follows that  $x'(v^*) < y'(v^*)$ , which is a contradiction to  $x(\tau) > y(\tau)$  for all  $\tau \in (v^*, 1]$ . Hence, we must have  $x(\tau) = y(\tau)$  for all  $\tau \in (v^*, 1]$  in this case. Second, if  $k(v, c) > 0$  with zero probability measure on  $(v^*, 1) \times (c(v^*), 1)$ , then, by the same argument,  $x'(\tau) = y'(\tau)$  for all  $\tau \in (v^*, 1]$ . It follows from  $x(v^*) = y(v^*)$  that  $x(\tau) = y(\tau)$  for all  $\tau \in (v^*, 1]$  in this case. Hence, both cases lead to a contradiction to  $x(1) > y(1)$ . Therefore, we have established that there exists a  $v^* \in [0, 1)$ , such that  $x(\tau) = y(\tau)$  for all  $\tau \in [v^*, 1]$ . If  $v^* = 0$ , the proof is complete.

Now, consider an arbitrary closed interval  $[\alpha, \beta] \subset [0, 1]$ . Suppose  $x(\alpha) = y(\alpha)$  and  $x(\beta) = y(\beta)$ ; and, without loss of generality,  $x(\tau) > y(\tau)$  for all  $\tau \in (\alpha, \beta)$ . It follows from the argument above that  $x'(\tau) < y'(\tau)$  for  $\tau \in (\alpha, \beta)$ , which is in contradiction to  $x(\tau) > y(\tau)$  for all  $\tau \in (\alpha, \beta)$ . Hence, we have established that  $x(v) = y(v)$  for all  $v \in [0, 1]$ . Therefore, the symmetric equilibrium is unique. This completes the proof.

**Proof of Corollary 1**

We compare the expected payoffs for  $n$  and  $n + 1$  bidders as follows:

$$c_n^*(v) = \int_0^v \left[ 1 - \int_t^1 \int_0^{c_n^*(\tau)} k(c, \tau)dc d\tau \right]^{n-1} dt,$$

$$c_{n+1}^*(v) = \int_0^v \left[ 1 - \int_t^1 \int_0^{c_{n+1}^*(\tau)} k(c, \tau)dc d\tau \right]^n dt,$$

If  $c_{n+1}^*(v) \geq c_n^*(v)$ , then

$$\int_0^v \left[ 1 - \int_t^1 \int_0^{c_{n+1}^*(\tau)} k(c, \tau)dc d\tau \right]^n dt < \int_0^v \left[ 1 - \int_t^1 \int_0^{c_n^*(\tau)} k(c, \tau)dc d\tau \right]^{n-1} dt,$$

which contradicts the assumption that  $c_{n+1}^*(v) \geq c_n^*(v)$ , and thus  $c_{n+1}^*(v) < c_n^*(v)$ .

**Proof of Proposition 4**

First, notice that  $F_A = F_B$  implies  $f_A = f_B$ . Next, we establish that  $c_A^*(1) > c_B^*(1)$ . Suppose, by contradiction, that  $c_A^*(1) \leq c_B^*(1)$ . Then, there exists a  $v^* \in [0, 1)$  such that  $c_A^*(v) \leq c_B^*(v)$  for all  $v \in (v^*, 1]$ . This implies that

$$c_A^{*f}(v^*) = \left[ 1 - \int_{v^*}^1 f(\tau)G_A(c_A^*(\tau))d\tau \right]^{n-1},$$

$$c_B^{*f}(v^*) = \left[ 1 - \int_{v^*}^1 f(\tau)G_B(c_B^*(\tau))d\tau \right]^{n-1}.$$

Since  $G_A < G_B$  and  $c_A^*(v) \leq c_B^*(v)$  for all  $v \in (v^*, 1]$ , we have  $G_A(c_A^*(\cdot)) < G_B(c_B^*(\cdot))$ . It follows from that  $c_A^{*'}(v^*) > c_B^{*'}(v^*)$ . It follows that there exists a  $\bar{v}^* > v^*$  such that  $c_A^*(v) > c_B^*(v)$  for all  $v \in (v^*, \bar{v}^*)$ , leading to a contradiction.

Then, we show  $c_A^*(v) > c_B^*(v)$  for all  $v \in (0, 1]$ . Without loss of generality, assume there exists a  $v^* \in [0, 1)$  such that  $c_A^*(v^*) = c_B^*(v^*)$  and  $c_A^*(v) > c_B^*(v)$  for all  $v \in (v^*, 1]$ . This implies that  $c_A^{*'}(v^*) > c_B^{*'}(v^*)$ , which means:

$$\int_{v^*}^1 f(\tau)G_A(c_A^*(\tau))d\tau < \int_{v^*}^1 f(\tau)G_B(c_B^*(\tau))d\tau.$$

In addition, since  $c_A^{*'}(v^*) > c_B^{*'}(v^*)$ , there exists a  $\underline{v}^* < v^*$  such that  $c_A^*(v) > c_B^*(v)$  for all  $v \in (\underline{v}^*, v^*)$  and  $c_A^*(\underline{v}^*) = c_B^*(\underline{v}^*)$  by noticing  $c_A^*(0) = c_B^*(0)$ . This implies that  $c_A^{*'}(\underline{v}^*) > c_B^{*'}(\underline{v}^*)$ . To see this, notice that Since  $G_A < G_B$  and  $c_A^*(v) < c_B^*(v)$  for all  $v \in (\underline{v}^*, v^*)$ , we have  $G_A(c_A^*(\cdot)) < G_B(c_B^*(\cdot))$  for all  $v \in (\underline{v}^*, v^*)$ , meaning

$$\int_{\underline{v}^*}^{v^*} f(\tau)G_A(c_A^*(\tau))d\tau < \int_{\underline{v}^*}^{v^*} f(\tau)G_B(c_B^*(\tau))d\tau.$$

Hence, we have

$$\begin{aligned} c_A^{*'}(\underline{v}^*) &= \left[ 1 - \int_{\underline{v}^*}^1 f(\tau)G_A(c_A^*(\tau))d\tau \right]^{n-1} \\ &= \left[ 1 - \int_{\underline{v}^*}^{v^*} f(\tau)G_A(c_A^*(\tau))d\tau - \int_{v^*}^1 f(\tau)G_A(c_A^*(\tau))d\tau \right]^{n-1} \\ &> \left[ 1 - \int_{\underline{v}^*}^{v^*} f(\tau)G_B(c_B^*(\tau))d\tau - \int_{v^*}^1 f(\tau)G_B(c_B^*(\tau))d\tau \right]^{n-1} \\ &= \left[ 1 - \int_{\underline{v}^*}^1 f(\tau)G_B(c_B^*(\tau))d\tau \right]^{n-1} = c_B^{*'}(\underline{v}^*). \end{aligned}$$

This then implies that there exists a  $\tilde{v} \in (\underline{v}^*, v^*)$  such that  $c_A^*(v) > c_B^*(v)$  for all  $v \in (\underline{v}^*, \tilde{v}) \subset (\underline{v}^*, v^*)$ , resulting in a contradiction. Hence,  $c_A^*(v) > c_B^*(v)$  for all  $v \in (0, 1]$ . This completes the proof.

### Proof of Proposition 5

We prove this theorem by using the contraction mapping theorem. First, let  $C([0, 1])$  be the space of continuous functions  $\psi$  mapping from  $[0, 1]$  to  $[0, 1]^2$  with the sup norm on  $[0, 1]$ :

$$\|\psi\| = \sup_{v \in [0,1]} |\psi(v)|,$$

where where  $|\cdot|$  is a norm in  $R^n$  and for any  $x \in R^n$ ,  $|\cdot|$  is defined as  $|x| = \max_{i \in N} |x_i|$ .

Define

$$M = \{\psi \in C([0, 1]) : \|\psi\| \leq 1\}.$$

It is certainly that the normed space  $(M, \|\cdot\|)$  is a Banach space. Notice that  $\mathbf{c}^*(\cdot) = (c_1(\cdot), c_2(\cdot)) \in M$ .

Next, define a mapping  $h_i(t_i, \mathbf{c}^*(\cdot))$  from  $[0, 1] \times [0, 1]^2 \rightarrow [0, 1]$ , for all  $i \in \{1, 2\}$  as follows:

$$h_1(t_1, \mathbf{c}^*(\cdot)) = 1 - \int_{v_2(b_1(t_1))}^1 \int_0^{c_2^*(\tau)} k_2(c, \tau) dc d\tau$$

$$h_2(t_2, \mathbf{c}^*(\cdot)) = 1 - \int_{v_1(b_2(t_2))}^1 \int_0^{c_1^*(\tau)} k_1(c, \tau) dc d\tau$$

Define  $H(t, \mathbf{c}^*(\cdot)) : [0, 1]^2 \times [0, 1]^2 \rightarrow [0, 1]^2$  as follows:

$$H(t, \mathbf{c}^*(\cdot)) = (h_1(t_1, \mathbf{c}^*(\cdot)), h_2(t_2, \mathbf{c}^*(\cdot)))'$$

Then, we define a mapping  $T : M \rightarrow M$  as follows:

$$\begin{aligned} (T\mathbf{c}^*)(v) &= \int_0^v H(t, \mathbf{c}^*(\cdot)) dt \\ &= \left[ \int_0^v \left( 1 - \int_{v_2(b_1(t_1))}^1 \int_0^{c_2^*(\tau)} k_2(c, \tau) dc d\tau \right) dt_1 \right. \\ &\quad \left. \int_0^v \left( 1 - \int_{v_1(b_2(t_2))}^1 \int_0^{c_1^*(\tau)} k_1(c, \tau) dc d\tau \right) dt_2 \right] \\ &= \left[ v - \int_0^v \int_{v_2(b_1(t_1))}^1 \int_0^{c_2^*(\tau)} k_2(c, \tau) dc d\tau dt_1 \right. \\ &\quad \left. v - \int_0^v \int_{v_1(b_2(t_2))}^1 \int_0^{c_1^*(\tau)} k_1(c, \tau) dc d\tau dt_2 \right] \end{aligned}$$

Since  $g_i(\cdot)$  and  $f_i(\cdot)$ ,  $i \in \{1, 2\}$ , are integrable over both arguments and bidding functions are differentiable (hence continuous),  $h_i(t_i, \mathbf{c}^*(\cdot))$  and  $H(t, \mathbf{c}^*(\cdot))$  are continuous, for all  $i \in N$ .

It suffices to show that  $Tc^*$  is a contraction mapping. To show this, for any  $x, y \in M$ , where  $x = (x_1, x_2)'$  and  $y = (y_1, y_2)'$ , we have:

$$\begin{aligned} (Tx)(v) - (Ty)(v) &= \left[ \int_0^v \int_{v_2^1(b_1(t_1))}^1 \int_0^{y_2(\tau)} k_2(c, \tau) dc d\tau dt_1 - \int_0^v \int_{v_2^1(b_1(t_1))}^1 \int_0^{x_2(\tau)} k_2(c, \tau) dc d\tau dt_1 \right] \\ &\quad - \left[ \int_0^v \int_{v_1^1(b_2(t_2))}^1 \int_0^{y_1(\tau)} k_1(c, \tau) dc d\tau dt_2 - \int_0^v \int_{v_1^1(b_2(t_2))}^1 \int_0^{x_1(\tau)} k_1(c, \tau) dc d\tau dt_2 \right] \\ &= \left[ \int_0^v \left( \int_{v_2^1(b_1(t_1))}^1 \int_{x_2(\tau)}^{y_2(\tau)} k_2(c, \tau) dc d\tau \right) dt_1 \right. \\ &\quad \left. - \int_0^v \left( \int_{v_1^1(b_2(t_2))}^1 \int_{x_1(\tau)}^{y_1(\tau)} k_1(c, \tau) dc d\tau \right) dt_2 \right]. \end{aligned}$$

Hence, we have

$$\begin{aligned} \|(Tx)(v) - (Ty)(v)\| &= \sup_{v \in [0,1]} \max_{i \in \{1,2\}} \left| \int_0^v \int_{v_j(b_i(t_i))}^1 \int_{x_j(\tau)}^{y_j(\tau)} k_j(c, \tau) dc d\tau dt_i \right| \\ &\leq \sup_{v \in [0,1]} \max_{i \in \{1,2\}} \\ &\quad \int_0^v \left( \bar{k}_j(1 - v_j(b_i(t_i))) \sup_{s \in [0,1]} \max_{j \in \{1,2\}} |x_j(s) - y_j(s)| \right) dt_i \\ &= \|x(s) - y(s)\| \left( \sup_{v \in [0,1]} \max_{i \in \{1,2\}} \left( \int_0^v \bar{k}_j(1 - v_j(b_i(t_i))) dt_i \right) \right) \\ &\leq \|x(s) - y(s)\| \left( \max_{i \in \{1,2\}} \left( \int_0^1 \bar{k}_j(1 - v_j(b_i(t_i))) dt_i \right) \right) \\ &< \|x(s) - y(s)\|, \end{aligned}$$

where  $\bar{k}_j := \sup_{c,v \in [0,1]} k_i(c, v)$ . The last inequality holds because  $\sup_{c,v \in [0,1]} k_i(c, v) < \frac{1}{\int_0^1 [1 - v_i(b_i(v))] dv}$  for  $i, j \in \{1, 2\}$  and  $i \neq j$ . Therefore,  $T : M \rightarrow M$  is a contraction mapping. Since  $M$  is a complete metric space, it follows from the contraction mapping theorem that  $T$  has a unique fixed point in  $M$ . Hence, functional equation  $\mathbf{c}^* = T\mathbf{c}^*$  has a unique solution.

This completes the proof.

### Proof of Corollary 2

Suppose the valuations and participation costs are independently distributed, i.e.  $K_i(v_i, c_i) = F_i(v_i)G_i(c_i)$  for all  $i \in N$ . The most of the argument in the proof of Proposition 5 applies here. However, we could have a weaker sufficient condition. To see this, notice that



$$\begin{aligned}
 \|(Tx)(v) - (Ty)(v)\| &= \sup_{v \in [0,1]} \max_{i \in \{1,2\}} \left| \int_0^v \int_{v_j(b_i(t_i))}^1 (G_j(y_j(\tau)) - G_j(x_j(\tau))) f_j(\tau) d\tau dt_i \right| \\
 &\leq \sup_{v \in [0,1]} \max_{i \in \{1,2\}} \int_0^v \int_{v_j(b_i(t_i))}^1 |G_j(y_j(\tau)) - G_j(x_j(\tau))| f_j(\tau) d\tau dt_i \\
 &\leq \sup_{v \in [0,1]} \max_{i \in \{1,2\}} \left( \bar{g}_j \int_0^v \int_{v_j(b_i(t_i))}^1 |y_j(\tau) - x_j(\tau)| f_j(\tau) d\tau dt_i \right) \\
 &\leq \sup_{v \in [0,1]} \max_{i \in \{1,2\}} \left( \bar{g}_j \int_0^v \int_{v_j(b_i(t_i))}^1 \left( \sup_{s \in [0,1]} \max_{i \in \{1,2\}} \int_0^s |y_j(u) - x_j(u)| du \right) f_j(\tau) d\tau dt_i \right) \\
 &= \|x(s) - y(s)\| \left( \sup_{v \in [0,1]} \max_{i \in \{1,2\}} \left( \int_0^v \int_{v_j(b_i(t_i))}^1 \bar{g}_j f_j(\tau) d\tau dt_i \right) \right) \\
 &\leq \|x(s) - y(s)\| \left( \max_{i \in \{1,2\}} \left( \int_0^1 \int_{v_j(b_i(t_i))}^1 \bar{g}_j f_j(\tau) d\tau dt_i \right) \right) \\
 &= \|x(s) - y(s)\| \left( \max_{i \in \{1,2\}} \left( \bar{g}_j \int_0^1 [1 - F_j(v_j(b_i(t_i)))] dt_i \right) \right) \\
 &< \|x(s) - y(s)\|,
 \end{aligned}$$

where  $\bar{g}_j := \sup_{c \in [0,1]} g_j(c)$ . The last inequality holds because  $\sup_{c \in [0,1]} g_j(c) < \frac{1}{\int_0^1 [1 - F(v_j(b_i(v)))] dv}$  for  $i, j \in \{1, 2\}$  and  $i \neq j$ . Last, notice that the sufficient conditions when distribution functions are independent and when the sufficient condition in the general case can be rewritten as follows:

$$\begin{aligned}
 \int_0^1 \int_{v_j(b_i(v))}^1 \bar{g}_j f_j(\tau) d\tau dv &< 1, \\
 \int_0^1 \int_{v_j(b_i(v))}^1 \bar{k}_j d\tau dv &< 1.
 \end{aligned}$$

It follows that

$$\int_0^1 \int_{v_j(b_i(v))}^1 \bar{g}_j f_j(\tau) d\tau dv < \int_0^1 \int_{v_j(b_i(v))}^1 \bar{k}_j d\tau dv < 1.$$

The sufficient condition when distribution functions are independent holds when the sufficient condition in the general case holds.

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## Declarations

**Conflict of interest** All authors have no conflicts of interests.

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